Some Notes Related to Module EN0311 (Control Part)
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Contents

1 Matrix algebra

1.1 Basic definitions

1.1.1 Order

Definition

An $m \times n$ matrix has m rows and n columns. The matrix has **order** $r \times c$ (german: Höhe mal Breite).

Example

$$A = \begin{pmatrix} 1 & 8 \\ 7 & 2 \\ 3 & 4 \end{pmatrix}$$

Matrix A has **order** 3×2.

1.1.2 Elements

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Matrix A has order $m \times n$. a_{ij} represents the **element** in the ith row and jth column.

1.2 Addition, subtraction scalar- and matrix multiplication

1.2.1 Addition and subtraction

Definitions and properties

- Two matrices can be **added or subtracted** if they have the **same order**.
- Their **sum or difference** is found by **adding or subtracting corresponding elements**.
- Matrix addition is **commutative**: $A + B = B + A$
- Matrix addition is **associative**: $A + (B + C) = (A + B) + C$
- Matrix addition is **distributive** (see 1.2.2): $k(A + B) = kA + kB$, k scalar

Example(s)

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 3 & 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 4 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1+1 & 5+2 & -2+0 \\ 3+1 & 1+1 & 1+4 \end{pmatrix} = \begin{pmatrix} 1 & 7 & -2 \\ 2 & 2 & 5 \end{pmatrix} = B + A$$

1.2.2 Scalar multiplications

$$\text{if } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ then } kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}, \ k \text{ scalar}$$

1.2.3 Matrix multiplications

Definition and properties

- If A is $m \times n$ and B is $r \times s$ the **product** of AB can **only** be formed **if** $n = r$. The **product is** then a $m \times s$ matrix.
- If A is a $m \times p$ matrix and B is a $p \times n$ matrix then the product $C = AB$ is defined as the $m \times n$ matrix with elements:

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n$$

- Or you can say: To find c_{ij} you take the ith row of A and pair its elements with the jth column of B. The paired elements are multiplied together and added to form c_{ij}.
- $C = AB \rightarrow B$ has been **premultiplied** by A, or alternatively A has been **postmultiplied** by B.
- In general $AB \neq BA$ and so matrix multiplication is **not commutative**.
- Matrix multiplication is **associative** and so: $(AB)C = A(BC)$.

Example(s)

$$A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = (3 \times 2) \text{ and } B = \begin{pmatrix} g & h & i & j \\ k & l & m & n \end{pmatrix} = (2 \times 4)$$

Reminder: To find c_{ij} you take the ith row of A and pair its elements with the jth column of B. The paired elements are multiplied together and added to form c_{ij}.

$$C = \begin{pmatrix} c_{11} = ag + bk & c_{12} = ah + bl & c_{13} = ai + bm & c_{14} = aj + bn \\ c_{21} = cg + dk & c_{22} = ch + dl & c_{23} = ci + dm & c_{24} = cj + dn \\ c_{31} = eg + fk & c_{32} = eh + fl & c_{33} = ei + fm & c_{34} = ej + fn \end{pmatrix} = 3 \times 4$$

What happens when there are more than two multiplicands?

$$X = ABC$$

1. Calculate the product $U = BC$.
2. Calculate the product $X = AU$.

1.3 The transpose of a matrix

1.3.1 Definition and properties

If A is an arbitrary $m \times n$ matrix, a related matrix is the **transpose** of A, written A^T. The transpose A^T is found by **interchanging the rows and columns** of A. A^T is an $n \times m$ matrix.

- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$, k scalar
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$ (Watch out!)
- $(A^{-1})^T = (A^T)^{-1}$

1.3.2 Example(s)

$$A = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{pmatrix}$$

1.4 Special square matrices

1.4.1 Square matrices

Definition

A matrix which has the **same number of rows as columns** is called **square** matrix.

Example(s)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 3 \times 3 = m \times m$$

1.4.2 Diagonal matrices

Definition

A square matrix which has elements that are **zero everywhere exept of the leading diagonal** is called a **diagonal** matrix.

Example(s)

Matrices A and B are diagonal matrices.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} B = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

1.4.3 Identity matrices

Definition and property

Diagonal matrices which have **only ones on their leading diagonal** are called a **identity** matrices and are denoted by the letter I.

If A is a square matrix then $IA = A$

Example(s)

Matrices A and B are identity matrices.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1.4.4 Symmetric matrices

Definition and property

If a square matrix A **and** its transpose A^T **are identical**, then A is said to be a **symmetric** matrix.

A symmetric matrix is symmetrical about its leading diagonal

Example(s)

$$A = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 6 & 9 \\ 2 & 9 & 13 \end{pmatrix} = \begin{pmatrix} \ddots & -4 & 2 \\ -4 & \ddots & 9 \\ 2 & 9 & \ddots \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 6 & 9 \\ 2 & 9 & 13 \end{pmatrix} = A$$

1.5 Determinants, Minor and Cofactor

1.5.1 Definition and properties

The determinant of a **square** $n \times n$ matrix A is denoted by $\mathbf{det(A)}$ or $|\mathbf{A}|$. If we take a determinant and delete row i and column j then the determinant remaining is called **minor** M_{ij} of the element a_{ij}. In general we can take any row i (or column j) and evaluate an $n \times n$ determinant $|A|$ as:

$$|A| = \sum_{j=1}^{n} \underbrace{(-1)^{i+j}}_{\text{sign}} a_{ij} \underbrace{M_{ij}}_{\text{minor}}$$

With **cofactor** $A_{ij} = (-1)^{i+j} M_{ij}$ (appropriate sign $\times$ minor):

$$|A| = \sum_{j=1}^{n} a_{ij} A_{ij}$$

- $|A^T| = |A|$
- |AB|=|A||B|
- $|k| = k$, k scalar

- A square matrix A is said to be **non-singular** if $|\mathbf{A}| \neq \mathbf{0}$ and **singular** if $|\mathbf{A}| = \mathbf{0}$.
- If there is a linear combination (rows or columns) in the matrix then the determinant of the matrix is always zero.
- If the elements in the upper or lower triangle of the matrix are all zero then the determinant of the matrix is the product of the elements on the leading diagonal.

$(-1)^{i+j}$ can be depicted as:

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} (-1)^2 & (-1)^3 & (-1)^4 & \dots \\ (-1)^3 & (-1)^4 & (-1)^5 & \dots \\ (-1)^4 & (-1)^5 & (-1)^6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The determinant of 2×2 matrix is calculated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

1.5.2 Example(s)

Expansion along the first row $\rightarrow i = 1$

$$A = \begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{4} \\ 4 & 0 & 5 \\ -6 & 2 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A| = \det(A) = \begin{vmatrix} 2 & -1 & 4 \\ 4 & 0 & 5 \\ -6 & 2 & 3 \end{vmatrix}$$

With $i = 1$ and $n = 3$:

$$\begin{aligned} |A| &= \sum_{j=1}^{3} \underbrace{(-1)^{1+j}}_{\text{sign}} a_{1j} \underbrace{M_{1j}}_{\text{minor}} \\ &= (-1)^2 a_{11} M_{11} + (-1)^3 a_{12} M_{12} + (-1)^4 a_{13} M_{13} \\ &= \underbrace{a_{11}}_{2} M_{11} - \underbrace{a_{12}}_{-1} M_{12} + \underbrace{a_{13}}_{4} M_{13} \end{aligned}$$

The minor of $a_{11} = 2$ is:

$$M_{11} = \begin{vmatrix} \not{2} & \not{-1} & \not{4} \\ \not{4} & \mathbf{0} & \mathbf{5} \\ \not{-6} & \mathbf{2} & \mathbf{3} \end{vmatrix} = \begin{vmatrix} 0 & 5 \\ 2 & 3 \end{vmatrix} = (0 \times 3) - (5 \times 2) = -10$$

The minor of $a_{12} = -1$ is:

$$M_{12} = \begin{vmatrix} \not{2} & \not{-1} & \not{4} \\ \mathbf{4} & \not{0} & \mathbf{5} \\ \mathbf{-6} & \not{2} & \mathbf{3} \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ -6 & 3 \end{vmatrix} = (4 \times 3) - (5 \times -6) = 42$$

The minor of $a_{13} = -1$ is:

$$M_{13} = \begin{vmatrix} 4 & 0 \\ -6 & 2 \end{vmatrix} = (4 \times 2) - (0 \times -6) = 8$$

All together:

$$\begin{aligned} |A| &= 2 \times (-10) - (-1) \times 42 + 4 \times 8 \\ &= -20 + 42 + 32 \\ &= 54 \end{aligned}$$

Linear combination

The first two rows are a linear combination, row 1 multiplied by 2 is row 2. Therefore the determinant is zero!

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & -1 & 0 \end{vmatrix} = 0$$

All elements in the lower/upper triangle are zero

The determinant is the product of the leading diagonal.

$$\begin{vmatrix} \mathbf{1} & 0 & 0 & 0 \\ 2 & \mathbf{4} & 0 & 0 \\ 5 & 8 & \mathbf{9} & 0 \\ 20 & -4 & 3 & -1 \end{vmatrix} = 1 \times 4 \times 9 \times (-1) = -36$$

$$\begin{vmatrix} \mathbf{1} & 4 & 2 & 8 \\ 0 & \mathbf{4} & 5 & 6 \\ 0 & 0 & \mathbf{9} & 1 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1 \times 4 \times 9 \times (-1) = -36$$

1.6 Adjoint

1.6.1 Definition and properties

The adjoint of a square matrix A is the **transpose of the matrix of cofactors**, so for a 3×3 matrix A:

$$\text{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T$$

Reminder: Cofactor $A_{ij} = (-1)^{i+j} M_{ij}$ (appropriate sign $\times$ minor).

- $A(\text{adj}\, A) = |A|I$
- $\text{adj}(AB) = (\text{adj}\, B)(\text{adj}\, A)$

1.6.2 Example(s)

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Cofactors for the first row:

$$\begin{aligned}
A_{11} &= (-1)^{1+1}M_{11} = M_{11} \\
&= \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} = 1 \times 3 - 5 \times 2 \\
&= -7 \\
A_{12} &= (-1)^{1+2}M_{12} = -M_{12} \\
&= -\begin{vmatrix} 3 & 5 \\ -1 & 3 \end{vmatrix} = -(3 \times 3 - 5 \times (-1)) \\
&= -14 \\
A_{13} &= (-1)^{1+3}M_{13} = M_{13} \\
&= \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = 3 \times 2 - 1 \times (-1) \\
&= 7
\end{aligned}$$

Cofactors for the second row:

$$\begin{aligned}
A_{21} &= (-1)^{2+1}M_{21} = -M_{21} \\
&= -\begin{vmatrix} -2 & 0 \\ 2 & 3 \end{vmatrix} = -(-2 \times 3 - 0 \times 2) \\
&= 6 \\
A_{22} &= (-1)^{2+2}M_{22} = M_{22} \\
&= \begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} = 1 \times 3 - 0 \times -1 \\
&= 3 \\
A_{23} &= (-1)^{2+3}M_{23} = -M_{23} \\
&= -\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = -(1 \times 2 - (-2) \times (-1)) \\
&= 0
\end{aligned}$$

Cofactors for the third row:

$$\begin{aligned}
A_{31} &= (-1)^{3+1} M_{31} = M_{31} \\
&= \begin{vmatrix} -2 & 0 \\ 1 & 5 \end{vmatrix} = (-2) \times 5 - 0 \times 1 \\
&= -10 \\
A_{32} &= (-1)^{3+2} M_{32} = -M_{32} \\
&= -\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = -(1 \times 5 - 0 \times 3)) \\
&= -5 \\
A_{33} &= (-1)^{3+3} M_{33} = M_{33} \\
&= \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 1 \times 1 - (-2) \times 3 \\
&= 7
\end{aligned}$$

$$\text{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \begin{pmatrix} -7 & -14 & 7 \\ 6 & 3 & 0 \\ -10 & -5 & 7 \end{pmatrix}^T = \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

1.7 Inverse matrices

1.7.1 Definition and properties

The inverse of a matrix A is given by:

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \frac{\text{adj}(A)}{|A|}$$

- If A is **non-singular** (i. e. $|A| \neq 0$) the inverse of A exists.
- If A is **singular** (i. e. $|A| = 0$) the inverse of A **does not** exist.
- $(AB)^{-1} = B^{-1}A^{-1}$

1.8 Eigenvalue

1.8.1 Definition and properties

Given a square matrix A, its eigenvalues are the **roots** of the following polynomial (**characteristic equation**). These are the values of λ for which $AX = \lambda X$ has no-trivial solutions.

$$p(\lambda) = \det(\lambda I - A)$$

- The denominator of the **transfer function** of a system is equal to the characteristic equation $p(\lambda) = \det(\lambda I - A)$.Therefore the poles of a transfer function are equal to the eigenvalues of the matrix A.

1.8.2 Example(s)

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$

$$\begin{aligned} p(\lambda) = |\lambda I - A| &= \left| \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \right| = \left| \begin{pmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{pmatrix} \right| \\ &= (\lambda - 1)(\lambda - 4) + 2 = \lambda^2 - 4\lambda - \lambda + 4 + 2 \\ &= \lambda^2 - 5\lambda + 6 \end{aligned}$$

Roots:

$$\begin{aligned} p(\lambda) &= \lambda^2 - 5\lambda + 6 = 0 \\ &\rightarrow \lambda_1 = 3 \\ &\rightarrow \lambda_2 = 2 \\ &\rightarrow p(\lambda) = (\lambda - 3)(\lambda - 2) \end{aligned}$$

Eigenvalues of A are 3 and 2.

2 Linear equations

A system of simultaneous linear equations can be written in matrix notation.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\quad \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n &= b_n \end{aligned}$$

Or:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{2n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Or:

$$Ax = b$$

- If $b = 0$ the equations are called **homogeneous**.
- If $b \neq 0$ the equations are called **non-homogeneous/inhomogeneous**.
- Cases:
  1. $b \neq 0$ and $|A| \neq 0$: One solution $x = A^{-1}b$.
  2. $b = 0$ and $|A| \neq 0$: Trivial solution $x = 0$.
  3. $b \neq 0$ and $|A| = 0$: No solution **or** infinitely many solutions.
  4. $b = 0$ and $|A| = 0$: Infinitely many solutions.

3 Laplace transformation

3.1 Definition

$$G(s) = \int_0^\infty g(t) \cdot e^{-st} dt = \mathcal{L}\{g(t)\} \quad (1)$$

With s as a complex variable and $g(t) = 0$ for $t < 0$!

3.2 A few important properties

If

$$F(s) = \mathcal{L}\{f(t)\}$$

then

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

(with $f(0)$ =initial condition) and

$$\mathcal{L}\left\{\int f(t)dt\right\}.$$

Finally,

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s)$$

if the initial conditions are zero. When we calculate the transfer function of a system we always assume the initial conditions to be zero!

3.3 Table of important transforms

With $n \in \mathbb{N}$ and $a \in \mathbb{C}$. Furthermore, if not noted differently, Re(s)> 0.

$g(t)$ with $t > 0$	$G(s)$
1 (step function)	$\frac{1}{s}$
$\delta(t)$ (Dirac function)	1
e^{at}	$\frac{1}{s-a}$ für Re(s) > Re(a)
t^n	$\frac{n!}{s^{(n+1)}}$
$t^n \cdot e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ für Re(s) > Re(a)
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$t \cdot \sin(\omega t)$	$\frac{2\omega s}{\left(s^2+\omega^2\right)^2}$
$t \cdot \cos(\omega t)$	$\frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$
$\frac{k\omega_0^2}{s^2+2D\omega_0 \cdot s+\omega_0^2}$	$\frac{k \cdot \omega_0 \cdot e^{-D\omega_0 t}}{\sqrt{1-D^2}} \cdot \sin\left(\omega_0 t\sqrt{1-D^2}\right) \quad 0 \leq D < 1$
$\frac{k\omega_0^2}{s(s^2+2D\omega_0 \cdot s+\omega_0^2)}$	$k\left[1-e^{-D\omega_0 t}\left(\cos\left(\omega_0 t\sqrt{1-D^2}\right)+\frac{D}{\sqrt{1-D^2}}\sin\left(\omega_0 t\sqrt{1-D^2}\right)\right)\right]$ $0 \leq D < 1$

4 State space

4.1 How to transform a nth order differential equation into a set of first order differential equations

Consider the nth order differential equation (input-output relationship/description):

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t) \tag{2}$$

With: $a_n, a_{n-1}, \ldots, a_0$ =constant (parameters)
→ LTI-System (Linear Time Invariant)

Since the system is of order n, we need n states variables: $x_1, x_2, \ldots, x_n$

The question is: Wow to we define $x_1, x_2, \ldots, x_n$? We set:

$$\begin{aligned}
x_1(t) &= y(t) \\
x_2(t) &= \frac{d^2 y(t)}{dt^2} = \dot{y}(t) = \dot{x}_1(t) \\
x_3(t) &= \frac{d^3 y(t)}{dt^3} = \ddot{y}(t) = \dot{x}_2(t) \\
x_4(t) &= \frac{d^4 y(t)}{dt^4} = \dddot{y}(t) = \dot{x}_3(t) \\
x_5(t) &= \frac{d^5 y(t)}{dt^5} = y^{(4)}(t) = \dot{x}_4(t) \\
&\vdots \\
x_{n-1}(t) &= \frac{d^{n-2} y(t)}{dt^{n-2}} = y^{(n-2)}(t) = \dot{x}_{(n-2)}(t) \\
x_n(t) &= \frac{d^{n-1} y(t)}{dt^{n-1}} = y^{(n-1)}(t) = \dot{x}_{(n-1)}(t)
\end{aligned}$$

So far we have:

$$\begin{aligned}
y(t) &= x_1(t) \\
x(t) &= \begin{pmatrix} x_1(t) \\ x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}
\end{aligned}$$

What we need is: $\dot{x}(t)$. Let's have a look on (2), for the sake of simplicity/clearness (t) is left out:

$$a_n \underbrace{y^{(n)}}_{\dot{x}_n} + a_{n-1} \underbrace{y^{(n-1)}}_{x_n=\dot{x}_{n-1}} + \cdots + a_2 \underbrace{\ddot{y}}_{x_3=\dot{x}_2} + a_1 \underbrace{\dot{y}}_{x_2=\dot{x}_1} + a_0 \underbrace{y^{(0)}}_{x_1} = b_0 u$$

Now we can write for the highest derivation $\rightarrow y^{(n)} = \dot{x}_n$ (which is not a state):

$$y^n(t) = -\frac{a_0}{a_n}y(t) - \frac{a_1}{a_n}\dot{y}(t) - \frac{a_2}{a_n}\ddot{y}(t) - \cdots - \frac{a_{n-1}}{a_n}y^{(n-1)}(t) + \frac{b_0}{a_n}u(t) \tag{3}$$

Or:

$$\dot{x}_n(t) = -\frac{a_0}{a_n}x_1(t) - \frac{a_1}{a_n}x_2(t) - \frac{a_2}{a_n}x_3(t) - \cdots - \frac{a_{n-1}}{a_n}x_n(t) + \frac{b_0}{a_n}u(t) \tag{4}$$

In Summary we have:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
\dot{x}_3(t) &= x_4(t) \\
&\vdots \\
\dot{x}_{n-1}(t) &= x_n(t) \\
\dot{x}_n(t) &= -\frac{a_0}{a_n}x_1(t) - \frac{a_1}{a_n}x_2(t) - \frac{a_2}{a_n}x_3(t) - \cdots - \frac{a_{n-1}}{a_n}x_n(t) + \frac{b_0}{a_n}u(t) \\
y(t) &= x_1(t)
\end{aligned}$$

Or in state space form, again, for the sake of simplicity/clearness (t) is left out:

$$\overbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix}}^{\dot{x}} = \overbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix}}^{A \text{ (A is of the \textit{companion} form (german: Kardinalform).)}} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}}^{x} + \overbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{b_0}{a_n} \end{pmatrix}}^{B} u$$

$$y = \overbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}}^{C} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}}^{x}$$

4.2 Solution of a linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$
$$x \in \mathbb{R}^n; u \in \mathbb{R} \text{ valid mostly for single input systems}$$

First of all note:

$$X(s) = (sI - A)^{-1}x(0) + \underbrace{(sI - A)^{-1}BU(s)}_{\text{convolution}}$$

By taking the Laplace inverse of the above equation, we get:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{5}$$

With:

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

In more general term (for general initial conditions),

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{6}$$

4.3 Transformation from state-space to transfer function

$$\sum \begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) \end{cases}$$
$$x \in \mathbb{R}^n; u \in \mathbb{R} \text{ valid mostly for single input systems}$$

By taking the Laplace transform of system ($\sum$) we get:

$$\mathcal{L}\left\{\dot{x}(t)\right\} = sX(s) - x(0)$$

With:

$$X(s) = \mathcal{L}\left\{x(t)\right\} = \begin{pmatrix} \mathcal{L}\left\{x_1(t)\right\} \\ \mathcal{L}\left\{x_2(t)\right\} \\ \vdots \end{pmatrix} = \begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \end{pmatrix}$$

Hence:

$$sX(s) - x(0) = AX(s) + BU(s) \tag{7}$$
$$Y(s) = CX(s) \tag{8}$$

From (7), we have:

$$\begin{aligned} sX(s) - AX(s) &= x(0) + BU(s) \\ (sI - A)X(s) &= x(0) + BU(s) \\ X(s) &= (sI - A)^{-1}(x(0) + BU(s)) \end{aligned}$$

From (8), we have:

$$Y(s) = CX(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}BU(s) \tag{9}$$

When we calculate a transfer function we always assume $x(0) = 0$. Hence,

$$Y(s) = C(sI - A)^{-1}BU(s)$$

That is:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\text{output}}{\text{input}} = C(sI - A)^{-1}B \tag{10}$$

Reminder: $U(s)$ is not a matrix because $u \in \mathbb{R}$ (single input system).

5 From the input-output relationship to the transfer function

Let

$$Y(s) = \mathcal{L}\{y(t)\}$$

and

$$U(s) = \mathcal{L}\{u(t)\}.$$

Then taking the Laplace transform of the input-output relationship (2) given above, we have:

$$a_n s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_2 s^2 Y(s) + a_1 sY(s) + a_0 Y(s) = b_0 U(s)$$

Or:

$$Y(s)[a_n s^n + \cdots + a_0] = b_0 U(s)$$

Therefore:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0}$$